Final Exam — Complex Analysis

Martini Plaza, Monday 26 January 2015, 14:00 - 17:00

Duration: 3 hours

Instructions

1. The test consists of 6 questions; answer all of them.
2. Each question gets 15 points and the number of points for each subquestion is indicated at the beginning of the subquestion. 10 points are "free" and the total number of points is divided by 10. The final grade will be between 1 and 10.
3. The use of books, notes, and calculators is not allowed.

Question 1 (15 points)

Consider the function $f(z) = ze^z$ with z in $\mathbb{C}$.

a. (7 points) Write $f(z)$ in the form $u(x,y) + iv(x,y)$ where $z = x + iy$.

b. (8 points) Use the Cauchy-Riemann equations to show that $f(z)$ is entire.

Solution

a. We have

$$f(z) = (x+iy)e^{x+iy} = (x+iy)e^x(\cos y + i\sin y) = e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y).$$

Therefore

$$u(x,y) = e^x(x\cos y - y\sin y), \quad v(x,y) = e^x(y\cos y + x\sin y).$$

b. We check that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We have

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y + \cos y),$$

and

$$\frac{\partial v}{\partial y} = e^x(x\cos y - y\sin y + \cos y) = \frac{\partial u}{\partial x}.$$

Furthermore,

$$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y),$$

and

$$\frac{\partial v}{\partial x} = e^x(y\cos y + x\sin y + \sin y) = -\frac{\partial u}{\partial y}.$$

Therefore the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous on $\mathbb{C}$ so f is analytic on $\mathbb{C}$, that is, entire.

Question 2 (15 points)

Consider the function $f(z) = ze^{1/z^2}$.

a. (9 points) Find the Laurent series for $f(z)$ in $|z| > 0$.

b. (6 points) What is the type of the singularity of $f(z)$ at 0? Explain your answer.

Solution

a. The Taylor series for e^w is

$$e^w = \sum_{j=0}^{\infty} \frac{w^j}{j!} = 1 + w + \frac{w^2}{2} + \frac{w^3}{6} + \cdots$$

and converges for all finite $|w|$. Therefore for all $|z| > 0$ we have

$$e^{1/z^2} = \sum_{j=0}^{\infty} \frac{1}{j!\, z^{2j}} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{6z^6} + \cdots$$

and the Laurent series of f is

$$ze^{1/z^2} = z\sum_{j=0}^{\infty} \frac{1}{j!\, z^{2j}} = \sum_{j=0}^{\infty} \frac{1}{j!\, z^{2j-1}} = z + \frac{1}{z} + \frac{1}{2z^3} + \frac{1}{6z^5} + \cdots$$

b. The Laurent series of f at 0 contains infinitely many negative powers, therefore 0 is an *essential singularity*.

Question 3 (15 points)

Consider the function

$$f(z) = \frac{e^{-iz}}{z^2+9}.$$

a. (6 points) Compute the residue of $f(z)$ at each one of the singularities of the function.

b. (9 points) Evaluate

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+9}\, dx.$$

Solution

a. The given function

$$f(z) = \frac{e^{-iz}}{z^2+9} = \frac{e^{-iz}}{(z-3i)(z+3i)}.$$

has singularities at $z = \pm 3i$. We have

$$\text{Res}(f; 3i) = \lim_{z\to 3i} (z-3i) f(z) = \lim_{z\to 3i} \frac{e^{-iz}}{z+3i} = \frac{e^3}{6i},$$

and

$$\text{Res}(f; -3i) = \lim_{z\to -3i} (z+3i) f(z) = \lim_{z\to -3i} \frac{e^{-iz}}{z-3i} = \frac{e^{-3}}{-6i}.$$

b. We have

$$\text{pv} \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+9}\, dx = \lim_{r\to\infty} I_r,$$

where

$$I_r = \int_{-r}^{r} \frac{e^{-ix}}{x^2+9}\, dx = \int_{\gamma_r} f(z)\, dz,$$

with $f(z)$ as in the previous subquestion and γ_r the straight line contour from $-r$ to r along the real axis.

We define the closed contour $\Gamma_r = \gamma_r + C_r^-$ where C_r^- is the semicircle $|z| = r$ in the lower half-plane going from r to $-r$. Therefore

$$\int_{\Gamma_r} f(z)\, dz = I_r + \int_{C_r^-} f(z)\, dz.$$

This implies

$$\lim_{r\to\infty} \int_{\Gamma_r} f(z)\, dz = \lim_{r\to\infty} I_r + \lim_{r\to\infty} \int_{C_r^-} f(z)\, dz = I + \lim_{r\to\infty} \int_{C_r^-} f(z)\, dz,$$

and

$$I = \lim_{r\to\infty} \int_{\Gamma_r} f(z)\, dz - \lim_{r\to\infty} \int_{C_r^-} f(z)\, dz.$$

For any $r > 3$ we have that $-3i$ is the only singularity of f inside Γ_r, therefore

$$\lim_{r\to\infty} \int_{\Gamma_r} f(z)\,dz = -2\pi i\,\mathrm{Res}(f, -3i) = -2\pi i \frac{e^{-3}}{-6i} = \frac{\pi e^{-3}}{3},$$

where we took into account that the contour Γ_r is negatively oriented.
From Jordan's lemma we also know that

$$\lim_{r\to\infty} \int_{C_r^-} \frac{e^{-iz}}{z^2+9}\,dz = 0.$$

Therefore

$$I = \frac{\pi e^{-3}}{3}.$$

Question 4 (15 points)

a. (7 points) Given the function $f(z) = e^{\sin z}(z-i)^2(z+2)(z-3i)$ compute the integral

$$\int_C \frac{f'(z)}{f(z)}\,dz,$$

where C is the positively oriented circular contour with $|z| = 5/2$.

b. (8 points) Use Rouché's theorem to show that the polynomial $P(z) = z^4 - \frac{3}{2}z^3 + 1$ has exactly 4 roots in the disk $|z| < 2$.

Solution

a. The function f is entire and is therefore analytic on and inside C. We can then apply the Argument Principle to obtain that

$$\int_C \frac{f'(z)}{f(z)}\,dz = 2\pi i N_0(f),$$

where $N_0(f)$ is the number of zeros of f inside C (counting multiplicities). The only zeros of f inside C are i with multiplicity 2, and -2 with multiplicity 1. Therefore $N_0(f) = 3$ and

$$\int_C \frac{f'(z)}{f(z)}\,dz = 6\pi i.$$

b. Consider the closed contour C given by $|z| = 2$ and the functions $f(z) = z^4$ and $h(z) = -\frac{3}{2}z^3 + 1$, so that $P(z) = f(z) + h(z)$.
Then on C we have

$$|f(z)| = |z|^4 = 2^4 = 16,$$

and

$$|h(z)| \leq \left|-\frac{3}{2}\right| |z|^3 + |1| = 13 < |f(z)|.$$

Both f and h are entire functions and in particular they are analytic on and inside C, and since $|f(z)| < h(z)$ on C, Rouché's theorem can be applied and we obtain

$$N_0(f) = N_0(P).$$

Furthermore, $N_0(f) = 4$ so $P(z)$ has 4 zeros inside C.

Question 5 (15 points)

We denote by $\operatorname{Log} z$ the principal value of the logarithm $\log z$.

a. (6 points) Give without proof the expression for $\operatorname{Log} z$ in terms of the absolute value and the principal argument of z. Where is $\operatorname{Log} z$ analytic?

b. (9 points) Show that the function $g(z) = \operatorname{Log}(-z) + i\pi$ is a branch of $\log z$. Where is $g(z)$ analytic?

Solution

a. The expression is

$$\operatorname{Log} z = \operatorname{Log} |z| + i \operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the principal argument.
$\operatorname{Log} z$ is anlaytic on the whole complex plane except the non-positive real axis. We call this domain D_π.

b. The function $g(z)$ is analytic exactly where $\operatorname{Log}(-z)$ is analytic. Since $\operatorname{Log} z$ is analytic on D_π (the whole complex plane except the non-positive real axis) we conclude that $\operatorname{Log}(-z)$ is analytic on the whole complex plane except the non-negative real axis. We call this domain D_0.
To show that $g(z)$ is a branch of $\log z$ in D_0 we must show that $g(z)$ is equal to some value of $\log z$ in D_0.
We have

$$g(z) = \operatorname{Log}(-z) + i\pi = \operatorname{Log} |-z| + i \operatorname{Arg}(-z) + i\pi = \operatorname{Log} |z| + i \operatorname{Arg}(-z) + i\pi.$$

If for $z \in D_0$ we have $\operatorname{Im} z \geq 0$ (so, if $0 < \operatorname{Arg} z \leq \pi$) then $\operatorname{Arg}(-z) = \operatorname{Arg} z - \pi$. On the other hand, if $\operatorname{Im} z < 0$ (so, if $-\pi < \operatorname{Arg} z < 0$) then $\operatorname{Arg}(-z) = \operatorname{Arg} z + \pi$. Therefore,

$$g(z) = \begin{cases} \operatorname{Log} |z| + i \operatorname{Arg} z, & \text{for } \operatorname{Im} z \geq 0, \\ \operatorname{Log} |z| + i \operatorname{Arg} z + 2\pi i, & \text{for } \operatorname{Im} z < 0. \end{cases}$$

In both cases, $g(z) = \log z = \operatorname{Log} |z| + i \arg z$ for some choice of $\arg z$. Therefore, $g(z)$ is a branch of $\log z$.

Question 6 (15 points)

a. (8 points) Prove that if a function $f(z)$ is analytic inside and on a circle C_R of radius R centered at z_0 and if $|f(z)| \leq M$ for all z on C_R, then

$$|f'(z_0)| \leq \frac{M}{R}.$$

[Hint: recall the generalized Cauchy integral formula]

b. (7 points) Use the estimate of $|f'(z_0)|$ from the previous subquestion to show that a bounded entire function must be constant.

Solution

a. From the generalized Cauchy integral formula we have

$$f'(z_0) = \frac{1}{2\pi i} \int_{C_R} \frac{f(z)}{(z - z_0)^2}\, dz.$$

On C_R we have

$$\left| \frac{f(z)}{(z - z_0)^2} \right| = \frac{|f(z)|}{|z - z_0|^2} = \frac{|f(z)|}{R^2} \leq \frac{M}{R^2}.$$

Therefore,

$$|f'(z_0)| \leq \frac{1}{|2\pi i|} \frac{M}{R^2} (2\pi R) = \frac{M}{R}.$$

b. Suppose that $f(z)$ is an entire function with $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Then for any $R > 0$ we have

$$|f'(z_0)| \leq \frac{M}{R}.$$

Since R can be taken arbitrarily large, $|f'(z_0)|$ is a non-negative number that is smaller than any positive number so it must be zero. Alternatively, suppose $|f'(z_0)| > 0$ and choose R large enough so that $M/R < |f'(z_0)|$ leading to a contradiction. Therefore, $f'(z_0) = 0$ for any $z_0 \in \mathbb{C}$ and $f(z)$ is constant.